Fourth Part: The Interplay of Algebra and Logic

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Admissibility of cut

Cut elimination, in proof-theoretic parlance, is *constructive*: it implies giving an algorithm for removing cuts from derivations. The admissibility of cut for the cut-free system, however, can be attained by non-constructive algebraic methods, originating in the work of Maehara and Okada. The cut-free system, in other words, is shown to be complete with respect to some class of algebras. Such results suffice for establishing decidability or interpolation.

The algebraic approach to completeness has made sequent calculi more attractive to algebraists, and promises to lead to insights and proofs unobtainable solely by syntactic means. For example, Ciabattoni, Galatos, and Terui have combined algebraic and syntactic techniques to obtain an algorithm that converts Hilbert-style axioms of a certain form into structural rules for sequent and hypersequent calculi that preserve cut elimination.

Admissibility of cut for GRL (1)

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a monoid, and for each $X, Y \subseteq M$, define:

$$\begin{aligned} X \cdot Y &= \{x \cdot y : x \in X \text{ and } y \in Y\}; \\ X \backslash Y &= \{y \in M : X \cdot \{y\} \subseteq Y\}; \\ Y / X &= \{y \in M : \{y\} \cdot X \subseteq Y\}. \end{aligned}$$

$\wp(\mathbf{M}) = \langle \wp(M), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$ is a residuated lattice.
A *nucleus* on the powerset $\wp(M)$ is a map $\gamma : \wp(M) \to \wp(M)$ satisfying $X \subseteq \gamma(X)$, $\gamma(\gamma(X)) \subseteq \gamma(X)$, $X \subseteq Y$ implies $\gamma(X) \subseteq \gamma(Y)$, and $\gamma(X) \cdot \gamma(Y) \subseteq \gamma(X \cdot Y)$.

Lemma

If $\mathbf{M}$ is a monoid and γ is a nucleus on $\wp(M)$, then

$$\wp(\mathbf{M})_\gamma = \langle \gamma(\wp(M)), \cap, \cup_\gamma, \cdot_\gamma, \backslash, /, \gamma(\{1\}) \rangle$$

is a complete residuated lattice with $X \cup_\gamma Y = \gamma(X \cup Y)$ and $X \cdot_\gamma Y = \gamma(X \cdot Y)$.

Admissibility of cut for GRL (2)

We construct a residuated lattice such that validity in this algebra corresponds to cut-free derivability in GRL. Let $\mathbf{Fm}^*$ be the free monoid generated by the formulas of GRL: the elements of $\mathbf{Fm}^*$ are finite sequences of formulas, multiplication is concatenation, and the unit element is the empty sequence. Intuitively, we build our algebra from sets of sequences of formulas that "play the same role" in cut-free derivations in GRL.
We define:

$$
\begin{aligned}
[\Gamma_1 \_ \Gamma_2 \Rightarrow \alpha] &= \{\Gamma \in Fm^* : \Gamma_1, \Gamma, \Gamma_2 \Rightarrow \alpha \text{ is cut-free derivable in } \mathrm{GRL}\};\\
\mathcal{D} &= \{[\Gamma_1 \_ \Gamma_2 \Rightarrow \alpha] : \Gamma_1, \Gamma_2 \in Fm^* \text{ and } \alpha \in Fm\};\\
\gamma(X) &= \bigcap\{Y \in \wp(Fm^*) : X \subseteq Y \subseteq \mathcal{D}\}.
\end{aligned}
$$

Then γ is a nucleus on $\wp(\mathbf{Fm}^*)$ and hence the algebra $\wp(\mathbf{Fm}^*)_\gamma$ is a residuated lattice.

Admissibility of cut for GRL (3)

We define an evaluation for this algebra by $e(p) = \gamma(\{p\})$ and prove by induction on formula complexity that for each $\alpha \in Fm$:

$$\alpha \in e(\alpha) \subseteq [\_ \Rightarrow \alpha].$$

Now, let $\alpha_1, ..., \alpha_n \Rightarrow \beta$ be such that $\mathcal{RL} \vDash \alpha_1 \cdot ... \cdot \alpha_n \leq \beta$. So in particular $\wp(\mathbf{Fm}^*)_\gamma \vDash \alpha_1 \cdot ... \cdot \alpha_n \leq \beta$, whence $e(\alpha_1) \cdot ... \cdot e(\alpha_n) \subseteq e(\beta)$. However, since $\alpha_i \in e(\alpha_i)$ for $i \leq n$ and $e(\beta) \subseteq [\_ \Rightarrow \beta]$,

$$\alpha_1, ..., \alpha_n \in [\_ \Rightarrow \beta]$$

which means that $\alpha_1, ..., \alpha_n \Rightarrow \beta$ is cut-free derivable in GRL.

Decision problems

	Syntactic methods	**Semantic methods**
Validity problem (Decidability of the equational theory)	cut elimination (perhaps for display or hypersequent calculi)	FMP
Consequence problem (Decidability of the quasiequational theory)		SFMP

SFMP and FEP

Given an algebra $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} : i \in I \rangle \rangle$ of any type and $B \subseteq A$, a *partial subalgebra* $\mathbf{B}$ of $\mathbf{A}$ is the partial algebra $\langle B, \langle f_i^{\mathbf{B}} : i \in I \rangle \rangle$ where for $i \in I$, k-ary f_i, and $b_1, \ldots, b_k \in B$,

$$f_i^{\mathbf{B}}(b_1, \ldots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1, \ldots, b_k) \text{ if } f_i^{\mathbf{A}}(b_1, \ldots, b_k) \in B \\ \text{undefined, otherwise.} \end{cases}$$

An *embedding* of a partial algebra $\mathbf{B}$ into an algebra $\mathbf{A}$ of the same type is a 1-1 map $\varphi : B \longrightarrow A$ such that
$\varphi(f_i^{\mathbf{B}}(b_1, \ldots, b_k)) = f_i^{\mathbf{A}}(\varphi(b_1), \ldots, \varphi(a_k))$ whenever $f_i^{\mathbf{B}}(b_1, \ldots, b_k)$ is defined.
A class $\mathcal{K}$ of algebras of the same type has the *finite embeddability property* (FEP for short) if every finite partial subalgebra of some member of $\mathcal{K}$ can be embedded into some finite member of $\mathcal{K}$.

For quasivarieties of finite type such as (quasi)varieties of residuated lattices, the FEP is equivalent to the SFMP.

The FEP for residuated lattices (1)

Heyting algebras: let **B** be a finite partial subalgebra of some $\mathbf{A} \in \mathcal{HA}$. The lattice **D** generated by $B \cup \{0, 1\}$ is a finitely generated distributive lattice, hence finite, even though this might not be true of the Heyting algebra finitely generated by **B**. Since meet is residuated in any finite distributive lattice, **D** can be made into a Heyting algebra. Moreover, residuals coincide and thus **B** can be embedded into this algebra.

A more complicated construction by Blok and Van Alten establishes the FEP for numerous subvarieties of $\mathcal{FL}$ obeying integrality or idempotency. In particular, this construction was used by Ono to establish the decidability of various semilinear varieties corresponding to fuzzy logics.

The FEP for residuated lattices (2)

For varieties of residuated lattices such as $\mathcal{RL}$ and $\mathcal{CRL}$ that lack integrality and idemptency, (versions of) the following algebra based on the integers provides a good candidate for a counterexample:

$$\mathbf{Z} = \langle \mathbb{Z}, \min, \max, +, \to, 0 \rangle,$$

where $x \to y = -x + y$. The quasi-equation

$$1 \leq x \ \ \& \ \ x \cdot y \approx 1 \quad \Rightarrow \quad x \approx 1.$$

holds in all finite residuated lattices, but fails in $\mathbf{Z}$. So the SFMP and the FEP fail for $\mathcal{RL}$ and $\mathcal{CRL}$.

Decidability: A synopsis

Variety	Name	Equational Theory	Universal Theory
Residuated lattices	$\mathcal{RL}$	decidable	undecidable [80]
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	decidable	undecidable [91]
Distributive $\mathcal{RL}$	$\mathcal{DRL}$	decidable [86]	undecidable [50]
Distributive $\mathcal{CRL}$	$\mathcal{CDRL}$	decidable [21]	undecidable [50]
Idempotent $\mathcal{CDRL}$	$\mathcal{CI}d\mathcal{DRL}$	undecidable [132]	undecidable [132]
Integral $\mathcal{RL}$	$\mathcal{IRL}$	decidable	decidable [14]
Integral $\mathcal{CRL}$	$\mathcal{CIRL}$	decidable	decidable [14]
Semilinear $\mathcal{RL}$	$\mathcal{S}em\mathcal{RL}$		
Semilinear $\mathcal{CRL}$	$\mathcal{CS}em\mathcal{RL}$		
MTL-algebras	$\mathcal{MTL}$	decidable [85]	decidable [85]
Cancellative $\mathcal{RL}$	$\mathcal{C}an\mathcal{RL}$		
Cancellative $\mathcal{CRL}$	$\mathcal{CC}an\mathcal{RL}$		
ℓ-groups	$\mathcal{LG}$	decidable [77]	undecidable [58]
MV-algebras	$\mathcal{MV}$	decidable	decidable
Abelian ℓ-groups	$\mathcal{A}b\mathcal{LG}$	decidable [76]	decidable [76]
Heyting algebras	$\mathcal{HA}$	decidable [54]	decidable [54]
Boolean algebras	$\mathcal{BA}$	decidable	decidable

Amalgamation and interpolation

A variety $\mathcal{V}$ has the *amalgamation property* AP if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ and embeddings i and j of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$, respectively, there exist $\mathbf{D} \in \mathcal{V}$ and embeddings h, k of $\mathbf{B}$ and $\mathbf{C}$, respectively, into $\mathbf{D}$ such that $h \circ i = k \circ j$.

Let us write $var(K)$ for the variables occurring in some expression (formula, equation, set of equations, etc.) K. A variety $\mathcal{V}$ is said to have the *deductive interpolation property* DIP if whenever $\Sigma \vdash_{Eq(\mathcal{V})} \varepsilon$, there exists a set of equations Π with $var(\Pi) \subseteq var(\Sigma) \cap var(\varepsilon)$ such that $\Sigma \vdash_{Eq(\mathcal{V})} \Pi$ and $\Pi \vdash_{Eq(\mathcal{V})} \varepsilon$.

For varieties with at least one nullary operation and all operations of finite arity, the *AP* and the *DIP* are equivalent in the presence of the *congruence extension property*. So, this equivalence holds for $\mathcal{CRL}$ and its subvarieties.

The Craig interpolation property

A logic L has the *Craig interpolation property* CIP if, whenever $\vdash_{\mathrm{L}} \alpha \to \beta$, there exists a formula γ with $var(\gamma) \subseteq var(\alpha) \cap var(\beta)$ such that $\vdash_{\mathrm{L}} \alpha \to \gamma$ and $\vdash_{\mathrm{L}} \gamma \to \beta$.

Relationships among these notions

Theorem

Suppose that $\mathcal{V} \leq \mathcal{CRL}$ *is an equivalent algebraic semantics for a logic* L*. Then:*

1. $\Gamma \cup \{\alpha\} \vdash_{\mathrm{L}} \beta$ *iff* $\Gamma \vdash_{\mathrm{L}} (\alpha \wedge 1)^n \rightarrow \beta$ *for some* n*;*
2. *If* L *has the CIP, then* $\mathcal{V}$ *has the DIP and hence the AP.*

Proof.

(1) Induction on the height of derivations.
(2) It suffices by algebraizability to prove the logical counterpart of the DIP for L. Suppose that $\Gamma \vdash_{\mathrm{L}} \alpha$. Then $\gamma_1 \wedge \ldots \wedge \gamma_n \vdash_{\mathrm{L}} \alpha$ for some $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ and by (1), $\vdash_{\mathrm{L}} (\gamma_1 \wedge \ldots \wedge \gamma_n \wedge 1)^n \rightarrow \alpha$ for some n. If L has the CIP, then $\vdash_{\mathrm{L}} (\gamma_1 \wedge \ldots \wedge \gamma_n \wedge 1)^n \rightarrow \gamma$ and $\vdash_{\mathrm{L}} \gamma \rightarrow \alpha$ for some formula γ with $var(\gamma) \subseteq var(\gamma_1 \wedge \ldots \wedge \gamma_n) \cap var(\alpha)$. But then, again by (1), $\{\gamma_1 \wedge \ldots \wedge \gamma_n\} \vdash_{\mathrm{L}} \gamma$ and $\{\gamma\} \vdash_{\mathrm{L}} \alpha$ as required. □

Amalgamation for commutative FL algebras (1)

Theorem

FL_e *has the CIP.*

Proof.

We consider a *multiset version* GFL_e^m of the sequent calculus GFL_e and prove the following statement:
If $\vdash_{\mathrm{GFL}_e^m} \Gamma, \Delta \Rightarrow \alpha$, then there is β with $var(\beta) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ such that $\vdash_{\mathrm{GFL}_e^m} \Gamma \Rightarrow \beta$ and $\vdash_{\mathrm{GFL}_e^m} \Delta, \beta \Rightarrow \alpha$.

We proceed by induction on the heigth of a cut-free derivation of $\Gamma, \Delta \Rightarrow \alpha$ in GFL_e^m. If $\Gamma, \Delta \Rightarrow \alpha$ is an instance of (ID), then:

1. if $\Gamma = (\alpha)$, $\Delta = ()$, let $\beta = \alpha$;
2. if $\Gamma = ()$, $\Delta = (\alpha)$, let $\beta = 1$.
   Cases of the axioms for 0 and 1: similar.

□

Amalgamation for commutative FL algebras (2)

Proof.

For the inductive step, we must consider the last application of a rule in d. Example: implication. Suppose first that α is $\alpha_1 \rightarrow \alpha_2$ and d ends with:

$$\dfrac{\dfrac{\vdots}{\Gamma, \Delta, \alpha_1 \Rightarrow \alpha_2}}{\Gamma, \Delta \Rightarrow \alpha_1 \rightarrow \alpha_2} \ (\Rightarrow\rightarrow)$$

By IH there is β with $var(\beta) \subseteq var(\Gamma) \cap var(\Delta, \alpha_1 \rightarrow \alpha_2)$ such that $\vdash_{\mathrm{GFL}_e^m} \Gamma \Rightarrow \beta$ and $\vdash_{\mathrm{GFL}_e^m} \Delta, \beta, \alpha_1 \Rightarrow \alpha_2$. So $\vdash_{\mathrm{GFL}_e^m} \Delta, \beta \Rightarrow \alpha_1 \rightarrow \alpha_2$ by $(\Rightarrow\rightarrow)$. □

Amalgamation for commutative FL algebras (3)

Proof.

If d ends with:

$$\dfrac{\dfrac{\vdots}{\Gamma_1, \Delta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Gamma_2, \Delta_2, \gamma_2 \Rightarrow \alpha}}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2, \Delta_1, \Delta_2 \Rightarrow \alpha} \ (\to\Rightarrow)$$

there are two subcases:

1. If Γ is Γ_1, Γ_2 and Δ is $\gamma_1 \to \gamma_2, \Delta_1, \Delta_2$, then by IH (twice) there are β_1, β_2 s.t. $\vdash_{\mathrm{GFL}_e^m} \Gamma_i \Rightarrow \beta_i$ $(i \leq 2)$, $\vdash_{\mathrm{GFL}_e^m} \Delta_1, \beta_1 \Rightarrow \gamma_1$ and $\vdash_{\mathrm{GFL}_e^m} \Delta_2, \gamma_2, \beta_2 \Rightarrow \alpha$ (variables OK). So, for $\beta = \beta_1 \cdot \beta_2$,

$$\dfrac{\dfrac{\vdots}{\Gamma_1 \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Gamma_2 \Rightarrow \beta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \beta_1 \cdot \beta_2} \ (\Rightarrow \cdot) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Delta_1, \beta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Delta_2, \gamma_2, \beta_2 \Rightarrow \alpha}}{\Delta_1, \Delta_2, \gamma_1 \to \gamma_2, \beta_1, \beta_2 \Rightarrow \alpha} \ (\to\Rightarrow)}{\Delta_1, \Delta_2, \gamma_1 \to \gamma_2, \beta_1 \cdot \beta_2 \Rightarrow \alpha} \ (\cdot \Rightarrow)$$

Amalgamation for commutative FL algebras (4)

Proof.

1. If Γ is $\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2$ and Δ is Δ_1, Δ_2, then we consider the derivable sequent $\Gamma_1, \Delta_1 \Rightarrow \gamma_1$ and by IH we get a formula β_1 with $var(\beta_1) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ s.t. $\vdash_{\mathrm{GFL}_e^m} \Delta_1 \Rightarrow \beta_1$, $\vdash_{\mathrm{GFL}_e^m} \Gamma_1, \beta_1 \Rightarrow \gamma_1$. Arguing similarly for $\Gamma_2, \Delta_2, \gamma_2 \Rightarrow \alpha$, we get β_2 with $var(\beta_2) \subseteq var(\Gamma) \cap var(\Delta, \alpha)$ s.t. $\vdash_{\mathrm{GFL}_e^m} \Gamma_2, \gamma_2 \Rightarrow \beta_2$ and $\vdash_{\mathrm{GFL}_e^m} \Delta_2, \beta_2 \Rightarrow \alpha$. So, for $\beta = \beta_1 \to \beta_2$,

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1, \beta_1 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Gamma_2, \gamma_2 \Rightarrow \beta_2}}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2, \beta_1 \Rightarrow \beta_2}\,(\to\Rightarrow)}{\Gamma_1, \Gamma_2, \gamma_1 \to \gamma_2 \Rightarrow \beta_1 \to \beta_2}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\vdots}{\Delta_1 \Rightarrow \beta_1} \quad \dfrac{\vdots}{\Delta_2, \beta_2 \Rightarrow \alpha}}{\Delta_1, \Delta_2, \beta_1 \to \beta_2 \Rightarrow \alpha}\,(\to\Rightarrow)$$

□

Amalgamation and interpolation: A synopsis

Variety	Name	CIP	DIP	AP
Residuated lattices	$\mathcal{RL}$	yes	?	?
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	yes	yes	yes
Integral $\mathcal{CRL}$	$\mathcal{CIRL}$	yes	yes	yes
Semilinear $\mathcal{CRL}$	$\mathcal{CSemRL}$	no	?	?
MTL-algebras	$\mathcal{MTL}$	no	?	?
ℓ-groups	$\mathcal{LG}$	no	no	no
MV-algebras	$\mathcal{MV}$	no	yes	yes
Abelian ℓ-groups	$\mathcal{AbLG}$	no	yes	yes
Heyting algebras	$\mathcal{HA}$	yes	yes	yes
Boolean algebras	$\mathcal{BA}$	yes	yes	yes